Approximation Techniques for Single Integrals and Multiple Integrals

An Honors Thesis/Creative Project (ID 499)

by

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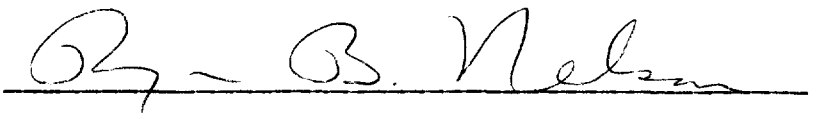
Thesis Director



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Introduction to Theory of Integration

Around the year 400 B.C., the Greek philosopher and mathematician Democritus found the volumes of a cone and a pyramid by treating each as a stack of thin layers. [3, p.37] The volume of each layer was determined by a relatively simple formula (length times width times height for a box, or πr^2 times height for a circular plate), and the sum of these accurately approximated the total volume of the object. During the centuries since then, many other situations have arisen where an area or a volume, or what it represented, needed to be determined. The technique Democritus used, an approximating sum whose limit was the actual volume, later developed into the Riemann integral. One of the shortcomings of this approach, however, was the length of work required to perform the calculations. This provided the impetus behind the development of a branch of mathematics which ultimately evolved into integral calculus. With integration, mathematicians had a simpler method of determining areas and volumes.

In the two-dimensional case, the area to be determined usually lies under a curve $f(x)$ and above the x axis, between two endpoints, a and b. An example of what this may represent physically is an ideal gas at constant temperature. If pressure is plotted along the y axis and volume along the x axis, pressure will decrease as volume increases according to the equation $PV=nRT$. [5, pp.64-65] (P is pressure, V is volume, n is the number of moles of gas, R is the gas

constant, and T is temperature.) This is represented in Figure 1.

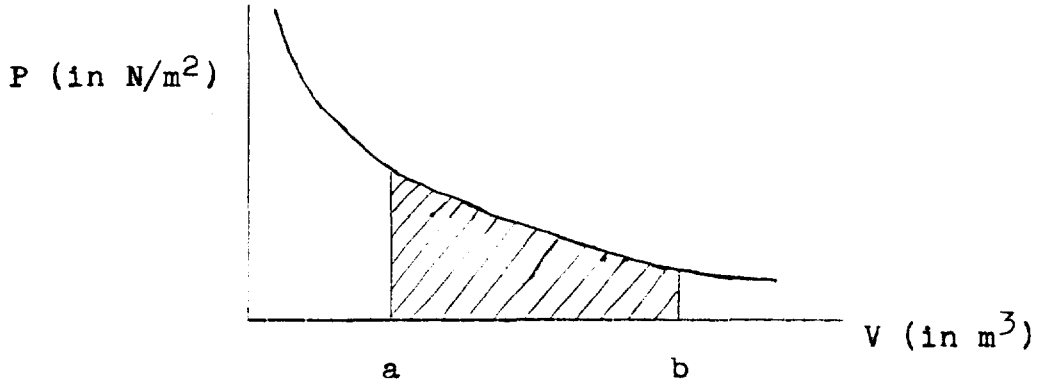


Figure 1

The area under the plotted curve represents the work done (in N·m) in changing the state of the system from one volume, a, to another volume, b.

A simple way to approximate the area under a curve involves splitting the interval from a to b into smaller subintervals of equal width. The area can then be approximated by using rectangles of equal width and summing their individual areas as shown in Figure 2.

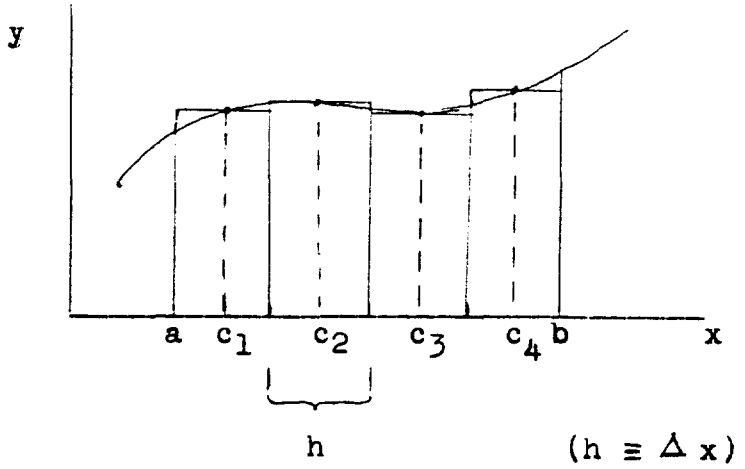


Figure 2

The points c_1 through c_4 are sampling points chosen within each subinterval to determine the height of each rectangle. If more rectangles are used, each with a smaller width h than before, the approximate area will be generally closer to the actual area. As the number of rectangles approaches infinity and the width h therefore approaches zero, the approximate area approaches the actual area. This can also be written as $\lim_{h\to 0} \sum_{i=1}^{n} f(c_i)\, h$ = actual area, where c_i is the sampling point in each subinterval.

In calculus, the exact area can often be found by evaluating the integral of f(x) from a to b ($\int_a^b f(x)\, dx$). This consists of finding an antiderivative of f(x), normally denoted F(x), such that the derivative of F(x) is f(x), and then determining F(b)-F(a). This is the fundamental theorem of calculus and is written

$$\int_a^b f(x)\, dx = F(b)-F(a).$$ [7, p.262]

The Problem

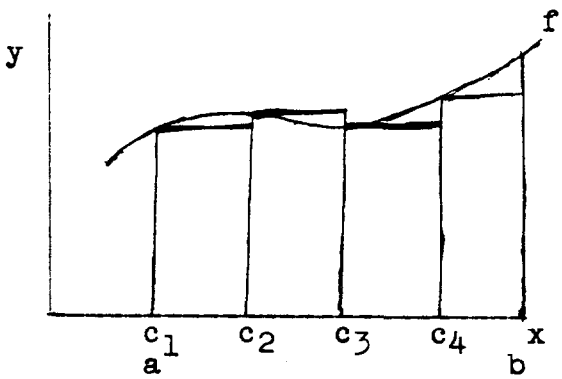
One can readily see that using the fundamental theorem of calculus to determine an area should be easier than summing the individual areas of many small rectangles. This is often true, but in some cases the antiderivative, $F(x)$, can not be found or can only be found after much tedious work.

Whenever the antiderivative does not exist or is very difficult to determine, certain techniques can be used to provide an approximation to the integral. All of these methods basically rely on a rectangular coordinate system, so a function in cylindrical or spherical coordinates can be expressed in rectangular coordinates by using the formulas below. The Jacobian is also listed for each case. Because it is introduced into the integral when changing from rectangular coordinates to another coordinate system, the Jacobian is dropped when changing back to rectangular coordinates.

cylindrical	spherical
$x=r\cos\theta$ $y=r\sin\theta$ $z=z$	$x=\rho\sin\phi\cos\theta$ $y=\rho\sin\phi\sin\theta$ $z=\rho\cos\phi$
Jacobian$=r$	Jacobian$=\rho^2\sin\phi$
	(ϕ= angle from z axis) (θ= angle from x axis)

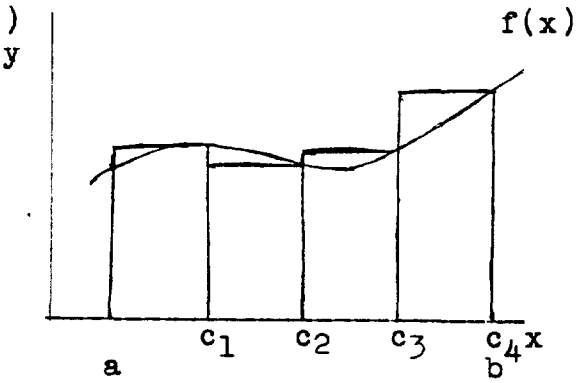
Approximation of Single Integrals

As mentioned in the introduction, a simple way to estimate a single integral is to use rectangles. There are basically two decisions to be made in using this method. The first is the choice of the sampling point, c_i. The sampling point can be chosen anywhere in the subinterval, including the left and right endpoints of the subinterval. As the subinterval width approaches zero (as the number of subintervals approaches infinity), however, the choice of the sampling point becomes meaningless. In Figure 3 and Figure 4, the left and right endpoints as sampling points are compared for four subintervals.



left endpoints

Figure 3



right endpoints

Figure 4

The second decision concerns the number of subintervals to be used. A more accurate estimate is given as more subintervals are used, regardless of the choice of sampling points, but more time is also required to perform the additional calculations.

With the rectangles, the height of each rectangle is basically a constant function f(x) over the subinterval.

Another method of approximating a single integral is to use a Taylor series. A Taylor series estimates the value of a function f(x) at a point x=a according to the formula $f(x)= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$. [6, p.110] For many cases, only the first few terms of the series are needed for a fairly accurate estimate of the function. The series can then be integrated term by term over the interval.

One of the problems that arises, however, is that the Taylor series approximation of the function is only accurate at or near the point a. If the interval of integration is very large, a Taylor series that is fairly accurate at both endpoints might not exist. [2, p.212] An example of this technique is shown below.

$$\int_1^{11} \ln x \, dx = 16.37685$$

$$\ln x = \ln a + \frac{1}{a}\frac{1}{1!}(x-a) - \frac{1}{a^2}\frac{1}{2!}(x-a)^2 + \ldots$$

(The value of a is chosen to be six because it is in the middle of the interval of integration.)

$$\ln x \approx \ln 6 + \frac{1}{6}(x-6) - \frac{1}{36}\frac{1}{2}(x-6)^2$$

$$\approx \ln 6 + \frac{x}{6} - 1 - \frac{(x^2-12x-36)}{72}$$

$$\approx \ln 6 + \frac{x}{6} - 1 - \frac{x^2}{72} + \frac{x}{6} - \frac{1}{2}$$

$$\approx \ln 6 - \frac{x^2}{72} + \frac{x}{3} - \frac{1}{2}$$

$$\int_1^{11} \ln x \, dx \approx \int_1^{11} \left(\ln 6 - \frac{x^2}{72} + \frac{x}{3} - \frac{1}{2} \right) dx$$

$$\int_1^{11} \ln x\,dx \approx \left(x \ln 6 - \frac{x^3}{216} + \frac{x^2}{6} - \frac{3}{2}x \right) \Bigg|_1^{11}$$

$$\int_1^{11} \ln x\,dx \approx 16.76019$$

(This approximation has an error of 2.34%.)

Yet another method of approximating single integrals involves fitting a polynomial to the actual curve. Whereas the Taylor series fitted a polynomial to the curve f(x) near one point, the Newton-Cotes method fits a polynomial to the curve over the entire interval. This is accomplished by providing "weights" which are used to multiply by the values of the function at equal intervals. [1, pp.284-286]

The Newton-Cotes method fits a polynomial of degree n to the curve y=f(x). The general form of a polynomial is $y=c_0x^n + c_1x^{n-1} + c_2x^{n-2} + \ldots + c_{n-1}x + c_n$, where n is the degree of the polynomial and c_0 through c_n are constants. In order to solve for the n+1 constants, at least n+1 equations are needed. This is achieved by dividing the interval from a to b into n subintervals, each of width h=(b-a)/n. Determining the constants provides a polynomial. Integrating this polynomial over the interval from a to b gives an approximate area.

The Newton-Cotes method carries this process a step further in providing weights for the y values for each polynomial of degree up to ten. [4, p.83] By summing each y value times its corresponding weight, and then multiplying

this sum by (b-a), an approximate area will be given. For a polynomial of degree one, the Newton-Cotes method reduces to the trapezoidal method, and for degree two it reduces to Simpson's method.

The weights for this method are given in Table 1, and some examples are given in Table 2. It is necessary to remember that each formula must use n intervals, or an integer times n. The weights are denoted C_k^n (where $0 \le k \le n$) and are determined by using $C_k^n = (1/n)\int_0^n L_k \, ds$, where

$$L_k = \frac{s(s-1)\cdots(s-k+1)(s-k-1)\cdots(s-n)}{k(k-1)\cdots(1)(-1)\cdots(k-n)} \text{ and } x = x_0 + sh. \quad [1, p.284]$$

An example of this is shown below.

$$C_1^2 = \frac{4}{6}$$

$$C_1^2 = \frac{1}{2}\int_0^2 L_1 \, ds$$

$$L_1 = \frac{s(s-2)}{(1)(-1)} = -(s^2 - 2s)$$

$$C_1^2 = -\frac{1}{2}\int_0^2 (s^2 - 2s)\, ds$$

$$= -\frac{1}{2}\left(\frac{s^3}{3} - s^2\right)\Bigg|_0^2$$

$$= -\frac{1}{2}\left(\frac{8}{3} - 4\right)$$

$$= \frac{1}{2}\left(4 - \frac{8}{3}\right)$$

$$= \frac{1}{2}\,\frac{4}{3}$$

$$C_1^2 = \frac{4}{6}$$

Table 1

degree n	C_0^n	C_1^n	C_2^n	C_3^n	C_4^n	C_5^n
1	$\frac{1}{2}$	$\frac{1}{2}$				
2	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$			
3	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$		
4	$\frac{7}{90}$	$\frac{32}{90}$	$\frac{12}{90}$	$\frac{32}{90}$	$\frac{7}{90}$	
5	$\frac{19}{288}$	$\frac{75}{288}$	$\frac{50}{288}$	$\frac{50}{288}$	$\frac{75}{288}$	$\frac{19}{288}$
6	$\frac{41}{840}$	$\frac{216}{840}$	$\frac{27}{840}$	$\frac{272}{840}$	$\frac{27}{840}$	$\frac{216}{840}$
7	$\frac{751}{17280}$	$\frac{3577}{17280}$	$\frac{1323}{17280}$	$\frac{2989}{17280}$	$\frac{2989}{17280}$	$\frac{1323}{17280}$
8	$\frac{989}{28350}$	$\frac{5888}{28350}$	$\frac{-928}{28350}$	$\frac{10496}{28350}$	$\frac{-4540}{28350}$	$\frac{10496}{28350}$
9	$\frac{2857}{89600}$	$\frac{15741}{89600}$	$\frac{1080}{89600}$	$\frac{19344}{89600}$	$\frac{5778}{89600}$	$\frac{5778}{89600}$
10	$\frac{16067}{598752}$	$\frac{106300}{598752}$	$\frac{-48525}{598752}$	$\frac{272400}{598752}$	$\frac{-260550}{598752}$	$\frac{427368}{598752}$

(Table 1 is continued on the next page for C_6^n through C_{10}^n.)

Table 1
(continued)

degree n	c_6^n	c_7^n	c_8^n	c_9^n	c_{10}^n
6	$\frac{41}{840}$				
7	$\frac{3577}{17280}$	$\frac{751}{17280}$			
8	$\frac{-928}{28350}$	$\frac{5888}{28350}$	$\frac{989}{28350}$		
9	$\frac{19344}{89600}$	$\frac{1080}{89600}$	$\frac{15741}{89600}$	$\frac{2857}{89600}$	
10	$\frac{-260550}{598752}$	$\frac{272400}{598752}$	$\frac{-48525}{598752}$	$\frac{106300}{598752}$	$\frac{16067}{598752}$

Table 2

$$\int_0^5 x^3\,dx = 156.25$$

n	number of intervals	approximation	% error
1	1	312.5	100.00
2	2	156.25	0.00
3	3	156.25	0.00
4	4	156.25	0.00
5	5	156.25	0.00
6	6	156.25	0.00
7	7	156.25	0.00
8	8	156.25	0.00
9	9	156.25	0.00
10	10	156.25	0.00

$$\int_0^5 x^{10}\,dx = 4{,}438{,}902.45$$

n	number of intervals	approximation	% error
1	1	24,414,062.5	450.00
2	2	8,169,809.99	84.05
3	3	6,421,358.62	44.66
4	4	4,781,782.63	7.72
5	5	4,638,781.26	4.50
6	6	4,453,794.25	0.34
7	7	4,448,078.12	0.21
8	8	4,439,024.89	0.00 (0.003)
9	9	4,438,987.37	0.00 (0.002)
10	10	4,438,920.46	0.00 (0.0004)

(Each approximation is rounded to five decimal places, and each percentage error is rounded to two decimal places.)

Table 2
(continued)

$$\int_1^{10} \frac{1}{x}\,dx = 2.30259$$

n	number of intervals	approximation	% error
1	1	4.95	114.98
2	2	2.74091	19.04
3	3	2.56339	11.33
4	4	2.38570	3.61
5	5	2.35982	2.49
6	6	2.32472	0.96
7	7	2.31876	0.70
8	8	2.30946	0.30
9	9	2.30778	0.23
10	10	2.30493	0.10

$$\int_0^{\pi/2} \cos x\,dx = 1$$

n	number of intervals	approximation	% error	
1	1	.78540	21.46	
2	2	1.00228	0.23	
3	3	1.00100	0.10	
4	4	.99999	0.00	(0.001)
5	5	.99999	0.00	(0.001)
6	6	1.00000	0.00	
7	7	1.00000	0.00	
8	8	1.00000	0.00	
9	9	1.00000	0.00	
10	10	1.00000	0.00	

Table 2
(continued)

$$\int_0^7 e^x \, dx = 1095.63316$$

n	number of intervals	approximation	% error	
1	1	3841.71605	250.64	
2	2	1435.11079	30.98	
3	3	1266.64821	15.61	
4	4	1117.12887	1.96	
5	5	1108.27742	1.15	
6	6	1096.80518	0.11	
7	7	1096.36952	0.07	
8	8	1095.68279	0.00	(0.005)
9	9	1095.66547	0.00	(0.003)
10	10	1095.63478	0.00	(0.0001)
1	10	1140.01036	4.05	
2	10	1097.01354	0.13	
5	10	1096.04740	0.04	
10	10	1095.63478	0.00	(0.0001)

When using the Newton-Cotes method, the amount of error involved is often important in determining the usefulness of the approximate answer given. Formulas exist for the maximum error under this method for polynomials up to degree six. [1, p.286] These formulas are listed in Table 3.

Table 3

n	Maximum Error
1	$\frac{8.3\times10^{-2}}{m^2}(b-a)^3 f^{(2)}(\xi)$
2	$\frac{3.5\times10^{-4}}{m^4}(b-a)^5 f^{(4)}(\xi)$
3	$\frac{1.6\times10^{-4}}{m^4}(b-a)^5 f^{(4)}(\xi)$
4	$\frac{5.2\times10^{-7}}{m^6}(b-a)^7 f^{(6)}(\xi)$
5	$\frac{3.0\times10^{-7}}{m^6}(b-a)^7 f^{(6)}(\xi)$
6	$\frac{6.4\times10^{-10}}{m^8}(b-a)^9 f^{(8)}(\xi)$

(ξ is that value of x in the interval from a to b which gives the maximum value of the derivative used in each formula. The number of subintervals is $m \cdot n$, where m is an integer and $m \geq 1$.)

If the derivative is too difficult to calculate directly, its value at a discrete set of points can be determined. This is done by dividing the interval from a to b into subintervals and calculating the slope of a line connecting the y values in each subinterval. This slope can then be plotted at a point between the endpoints of the subinterval on a

separate graph. This procedure can be repeated for higher derivatives. An illustration of this is shown in Figure 5.

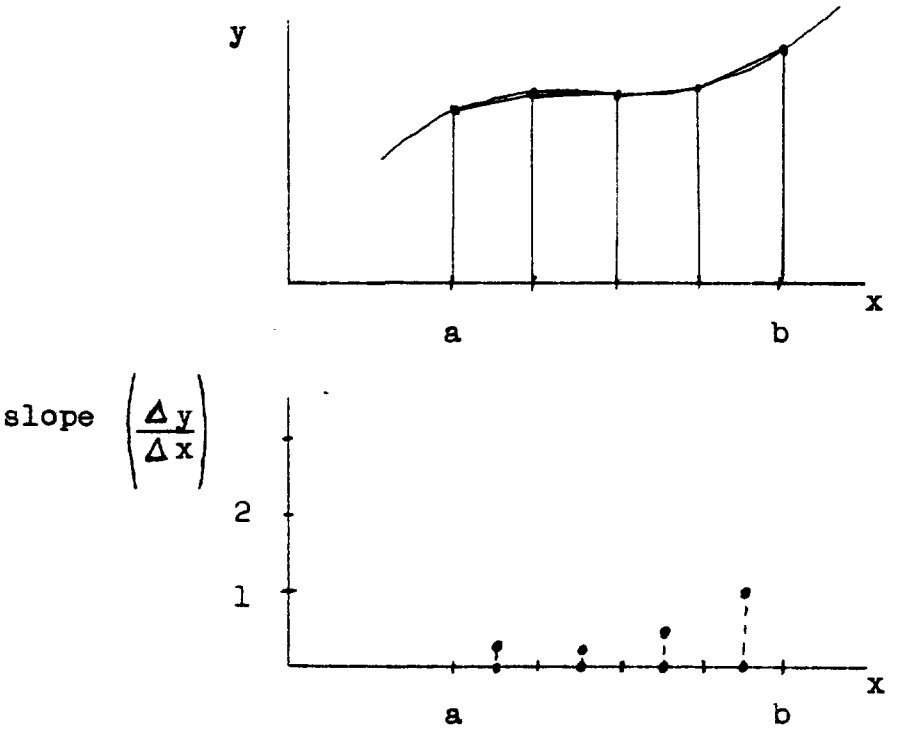


Figure 5

Approximation of Double Integrals

A double integral is normally used to determine the volume underneath a three-dimensional surface, or what this volume represents physically. The base underneath this surface is usually a square or a rectangle, or has two parallel sides with the two other sides represented as curves. These different bases are illustrated in Figure 6.

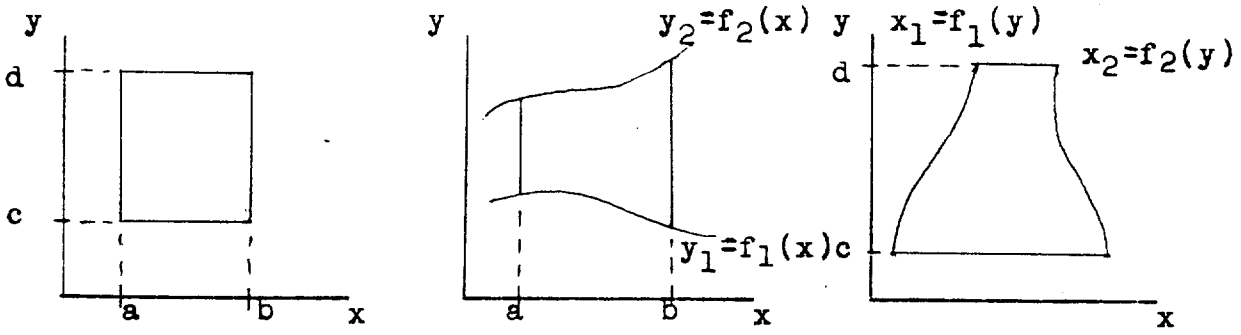


Figure 6

Because the three-dimensional surface is a function of both x and y, denoted f(x,y), it has to be integrated over both the x boundaries and the y boundaries. The resulting double integral is generally written as $\int_{x_1}^{x_2}\int_{y_1=f_1(x)}^{y_2=f_2(x)} f(x,y)\,dy\,dx$, or $\int_{y_1}^{y_2}\int_{x_1=f_1(y)}^{x_2=f_2(y)} f(x,y)\,dx\,dy$. In both cases the parallel boundaries are used for the outer integral.

When approximating double integrals, as well as triple integrals and other multiple integrals, the approximation techniques can be divided into two general categories. These categories are deterministic methods and stochastic

methods. Deterministic methods are direct calculations, whereas stochastic methods use random numbers and probability.

Perhaps the simplest deterministic method of approximating the volume is to use the length times width times height formula. By dividing the base into small regions, usually squares or rectangles, and then multiplying the area of each region times the height of a sampling point above it, the resulting sum provides an approximate volume. The accuracy of this method increases as the base is divided into smaller regions, but the number of calculations also increases.

Another way of approximating the double integral involves "slicing" the base parallel to the straight sides to produce a number of smaller regions. By using the Newton-Cotes method to determine the area of the resulting cross section, and then multiplying by the cross section width h, the approximate volume of each slice can be found. This is illustrated in Figure 7.

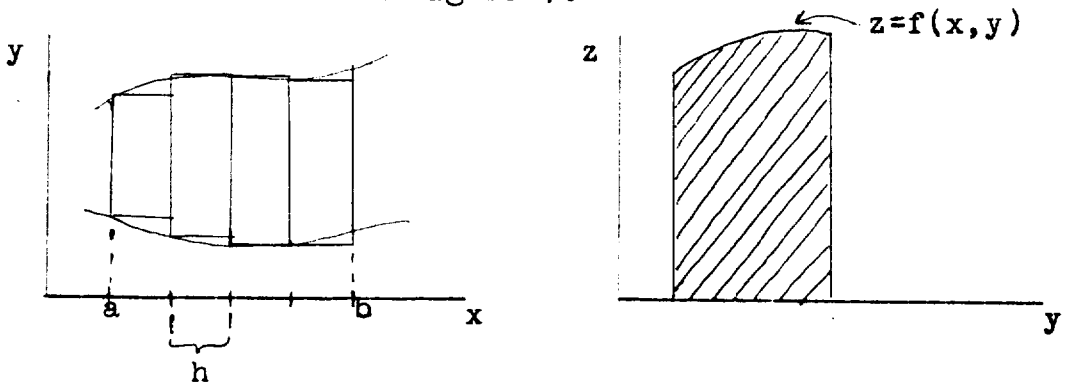


Figure 7

An example of this method is determining the volume of a loaf of sliced bread. Assuming each slice has width h, the area of one of the sides of the slice times the width h gives an approximate volume of the slice. Even though the two sides of a single slice frequently have different areas, the area of either side can be used to provide a fairly accurate volume of the slice. Summing the approximate volumes of the slices gives an approximate volume of the entire loaf. An example of this approximation technique is given in Table 4.

Table 4

$$\int_0^4 \int_0^4 x^5 y \, dy \, dx = 5461.33333$$

n	number of x intervals	number of y intervals	approximation	% error
1	1	10	14745.6	170
	1	100	16220.16	197
2	2	10	5529.6	1.25
	2	100	6082.56	11.38
3	3	10	5188.26667	5.00
	3	100	5707.09334	4.50
4	4	10	4915.20001	10.00
	4	100	5406.72001	1.00
5	5	10	4915.20001	10.00
	5	100	5406.72001	1.00
6	6	10	4915.20001	10.00
	6	100	5406.72002	1.00
7	7	10	4915.20001	10.00
	7	100	5406.72001	1.00
8	8	10	4915.20001	10.00
	8	100	5406.72001	1.00
9	9	10	4915.20001	10.00
	9	100	5406.72002	1.00
10	10	10	4915.20002	10.00
	10	100	5406.72002	1.00
1	10	10	5037.83425	7.75
2	10	10	4916.18305	9.98
5	10	10	4915.20001	10.00
10	10	10	4915.20002	10.00

Table 4
(continued)

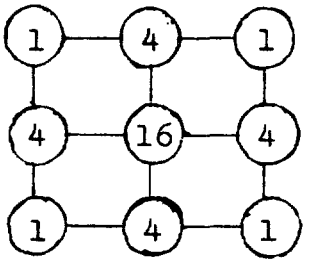
$$\int_0^4 \int_0^{x^2} x^5 y \, dy \, dx = 52428.8$$

n	number of x intervals	number of y intervals	approximation	% error
1	10	1	30110.1213	42.57
	100	1	49846.6779	4.93
2	10	2	30110.1213	42.57
	100	2	49846.6779	4.93
3	10	3	30110.1213	42.57
	100	3	49846.6779	4.93
4	10	4	30110.1213	42.57
	100	4	49846.6779	4.93
5	10	5	30110.1213	42.57
	100	5	49846.6779	4.93
6	10	6	30110.1213	42.57
	100	6	49846.6779	4.93
7	10	7	30110.1213	42.57
	100	7	49846.6779	4.93
8	10	8	30110.1213	42.57
	100	8	49846.6779	4.93
9	10	9	30110.1213	42.57
	100	9	49846.6779	4.93
10	10	10	30110.1213	42.57
	100	10	49846.6779	4.93
1	10	10	30110.1213	42.57
2	10	10	30110.1213	42.57
5	10	10	30110.1213	42.57
10	10	10	30110.1213	42.57
10	500	10	51906.0851	1.00

(Horizontal cross sections were used for the integral on page 19 and vertical cross sections were used for the integral on this page. The repetition in the approximations occurs when the degree of the polynomial being fitted to the cross section is higher than the degree of the actual curve.)

Whenever the volume to be found has a square base, there are two more techniques which can be used. Both of these methods involve weights which are multiplied by the z values over specific points on the base.

The first of these methods is Simpson's formula in two directions, which fits a surface of the form $z = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$ to the actual surface. The weights used in this method are shown in Figure 8 below. [1, p.299]

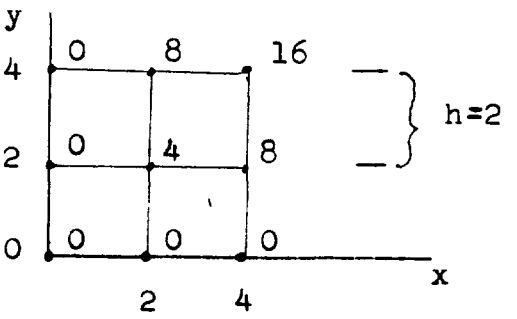


$$(9/h^2) \int\int f(x,y)\, dy\, dx$$

Figure 8

Each weight is multiplied by the corresponding z value as shown in the example in Figure 9.

$$\int_0^4 \int_0^4 xy\, dy\, dx = 64$$



$$(9/h^2) \int_0^4 \int_0^4 xy\, dy\, dx = 16(4) + 4(0+0+8+8) + 1(0+0+0+16)$$

$$\int_0^4 \int_0^4 xy\, dy\, dx = \frac{4}{9}(144) = 64$$

Figure 9

The second method is similar to Simpson's formula in two directions and was developed by Carl-Erik Fröberg. Fröberg fits an exponential surface to the actual function by defining a set of points in the region to be integrated as shown in Figure 10. [1, pp.296-299]

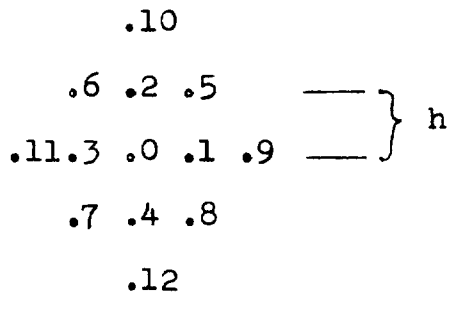
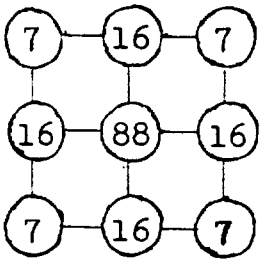


Figure 10

He then uses $\xi = h \frac{\partial}{\partial x}$ and $\eta = h \frac{\partial}{\partial y}$, along with the value for f_0, to state that $f_1 = e^{\xi} f_0$, $f_2 = e^{\eta} f_0$, $f_3 = e^{-\xi} f_0$, $f_4 = e^{-\eta} f_0$, $f_5 = e^{\xi+\eta} f_0$, and so on. The final result of this process is a set of weights which is shown in Figure 11. [1, p.299]

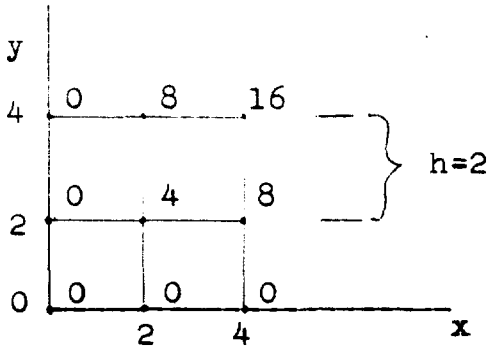


$$(45/h^2) \int\int f(x,y)\, dy\, dx$$

Figure 11

An example of Fröberg's method is shown in Figure 12.

$$\int_0^4 \int_0^4 xy \, dy \, dx = 64$$



$$(45/h^2) \int_0^4 \int_0^4 xy \, dy \, dx = 88(4) + 16(0+0+8+8) + 7(0+0+0+16)$$

$$\int_0^4 \int_0^4 xy \, dy \, dx = \frac{4}{45} (720) = 64$$

Figure 12

Both Simpson's formula in two directions and Fröberg's method can be used for bases of other shapes, as long as these shapes can be divided into square regions. By doing this, either of these two methods can be used to approximate the volume over each square region, and these can be summed over the entire base to approximate the total volume.

The final method which can be used to approximate double integrals is a stochastic approach known as the Monte Carlo method, and there are two variations of this method. Both of these involve generating random numbers, so a random number generator (on a computer or on a calculator) or a table of random numbers is necessary. Because of the nature of this technique, each time the Monte Carlo method is used it gives a different approximation. The average of these approximations, however, should be the actual answer or very close to it.

The first technique can best be visualized by using a single integral as an example. The area under the function $y = x^2$ from zero to three is nine square units. This is the same area as a square with a height of three units over the same interval. This is illustrated in Figure 13.

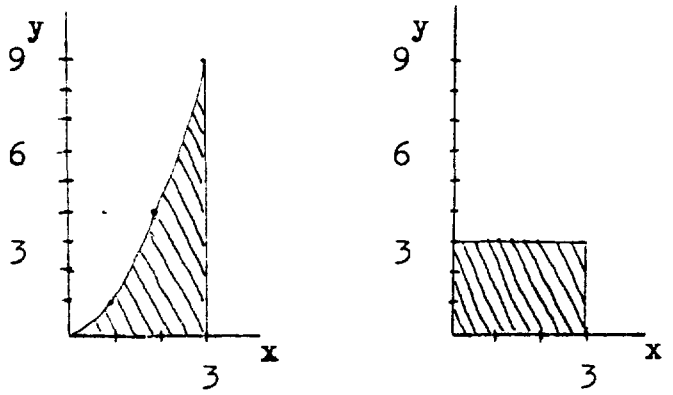


Figure 13

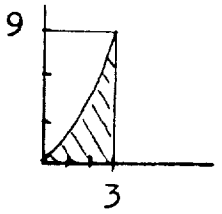
This illustration shows that the average value of $y = x^2$ over the interval from zero to three is three. The average value of a function is determined by dividing the value of the integral by the length of the interval of integration. [7, p.510] This average value of a function can be approximated and multiplied by the length of the interval to obtain an approximate area. The integral, I, is approximated according to the formula $I \approx \frac{V}{N} \sum_{i=1}^{N} f(\xi_i)$, where V is the volume of the base (or length of the interval in two dimensions), N is the number of random points generated, and $f(\xi_i)$ is the value of the function at that point. [8, p.193]

For the double integral $\int_0^4 \int_0^4 xy \, dy \, dx$, the base has an area of sixteen square units, and the random points generated must be between zero and four for x and y. The actual volume is 64, and the approximate volumes are shown in Table 5.

Table 5

number of random points (x,y)	approximate volume	% error
100	61.18682	4.40
1000	63.88973	0.17

The second Monte Carlo variation contains the actual function within a rectangle or square in two dimensions, or within a cube or box in three dimensions. Random numbers are then generated within this confinement region and are checked to see if they are within the actual function boundary. This formula for approximating the integral, I, is $I \approx \frac{n}{N} V$, where N is the number of random points generated, n is the number of these points within the function boundary, and V is the volume of the confinement region (or area in two dimensions). [1, p.275] For the single integral $\int_0^3 x^2 \, dx$, an illustration of the confinement region is shown in Figure 14.



confinement region:
$0 \leq x \leq 3$
$0 \leq y \leq 9$
area = 27

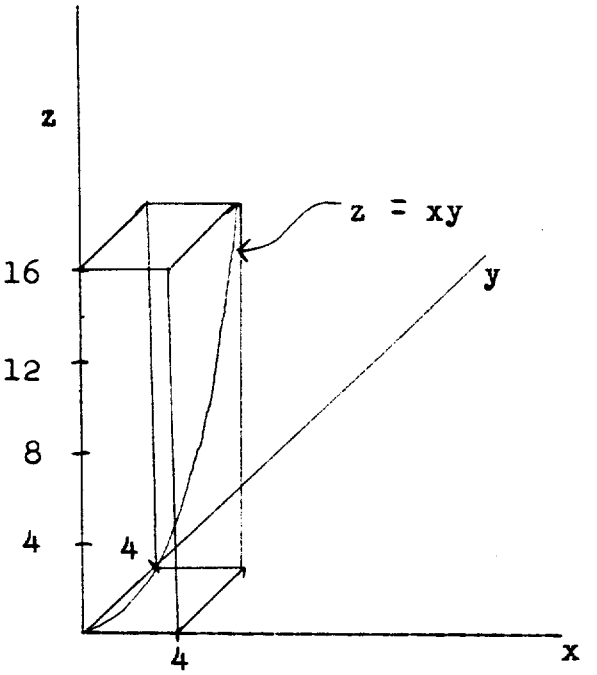
Figure 14

The random points are generated within the confinement region, and if one falls in the shaded region then n increases by one. A double integral example is shown in Figure 15.

$$\int_0^4 \int_0^4 xy \, dy \, dx = 64$$

confinement region

$V = 4 \cdot 4 \cdot 16 = 256$



number of random points (x,y,z)	approximate volume	% error
100	69.12	8.00
1000	65.536	2.40

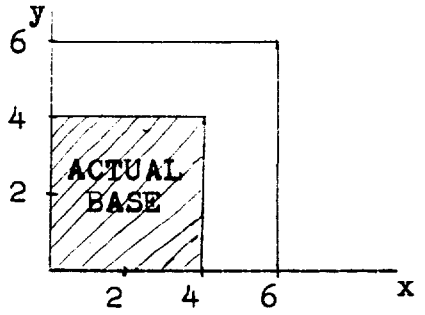
Figure 15

Also, the two variations of the Monte Carlo method can be combined to form $I \approx \frac{V}{N} \sum_{i=1}^{N} f(\xi_i)\, g(\xi_i)$, where V is the volume of a confinement region around the base, and $g(\xi_i)$ is a function defined as

$$g(\xi_i) = \begin{cases} 1 & \text{if } \xi_i \text{ is within the base} \\ 0 & \text{if } \xi_i \text{ is outside the base} \end{cases}$$

This formula is best suited for those cases where the value of the base of a function is difficult to determine, but where the base can be confined within a certain region. An example of this is shown in Figure 16.

$$\int_0^4 \int_0^4 xy \, dy \, dx = 64$$



confinement region

area = 6·6 = 36

number of random points (x,y)	approximate volume	% error
100	48.65314	23.98
1000	61.92941	3.24

Figure 16

With all of these variations in the Monte Carlo method, the accuracy increases as the number of random points used increases. Unfortunately, the time involved in performing the calculations also increases.

Approximation of Triple Integrals and Other Multiple Integrals

A triple integral is generally of the form $\iiint_A f(x,y,z)\,dV$, where the function is integrated over a specific volume in space. There are basically only two ways to approximate this type of integral.

The first method divides the volume in space into individual pieces of known volume. Within each piece a sampling point (x,y,z) is chosen and the value of the function at that point is determined. This function value is multiplied by the volume of the individual piece, and this process is repeated over the entire volume in space. The resulting sum is the approximate value of the integral.

The second technique is the Monte Carlo method which was introduced in the previous section. Using the formula $I \approx \frac{V}{N} \sum_{i=1}^{N} f(\xi_i)$, the average value of the function is approximated and multiplied by the volume of the "base" of the function. This base is the region of integration and has one less dimension than the function. Using the formula $I \approx \frac{n}{N} V$, V is the volume of the confinement region and has the same number of dimensions as the function. As random points within this confinement region are generated, each is checked to see if it falls within the boundaries of the function or not. The ratio of the points falling within the function boundaries to the total number of points generated is multiplied by the confinement region volume to approximate

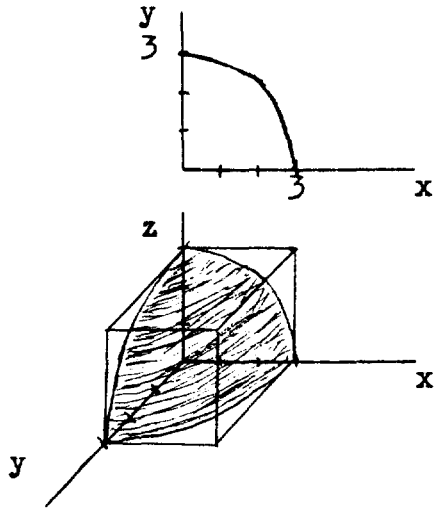
the integral.

For quadruple integrals and higher multiple integrals, the same methods used for triple integrals can also be used by increasing the number of dimensions accordingly. The most difficult part of this process is the visualization of a multidimensional function, and determining its "base" and confinement region for use in the variations of the Monte Carlo method. Once again, the important thing to remember is that the base has one less dimension than the function, and that the confinement region has the same number of dimensions as the function.

Examples of the Monte Carlo method for a triple integral and a quadruple integral are shown on the next two pages.

$$x^2 + y^2 + z^2 = 9$$

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{3} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{36\pi}{8} = 14.13717$$



base

area = $\frac{9\pi}{4}$ = 7.0658

confinement region

volume = 27

$$I \approx \frac{V}{N} \sum_{i=1}^{N} f(\xi_i) \qquad (V = \text{base volume})$$

number of random points (x,y)	approximation	% error
100	13.49584	4.54
1000	14.10361	0.24

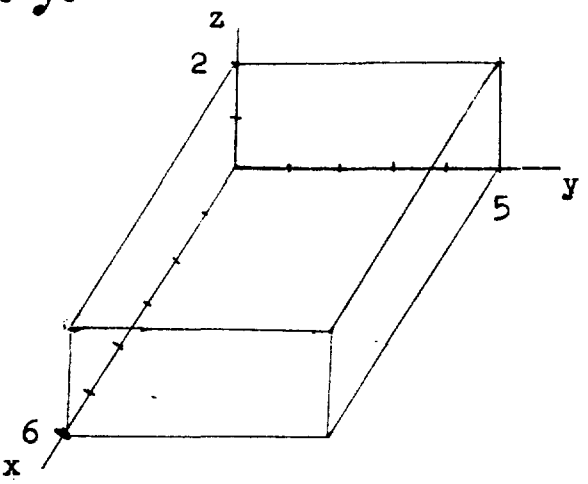
$$I \approx \frac{n}{N} V \qquad (V = \text{confinement region volume})$$

number of random points (x,y,z)	approximation	% error
100	13.77	2.60
1000	14.31	1.22

Figure 17

$q = xyz$

$$\int_0^6 \int_0^5 \int_0^2 \int_0^{xyz} dq\, dz\, dy\, dx = 450$$



base

volume = 60

confinement region = 60·60 = 3600

(maximum q value = 60 , base volume = 60)

$$I \approx \frac{V}{N} \sum_{i=1}^{N} f(\xi_i)$$

(V = base volume)

number of random points (x,y,z)	approximation	% error
100	438.29827	2.60
1000	458.23876	1.83

$$I \approx \frac{n}{N} V$$

(V = confinement region volume)

number of random points (x,y,z,q)	approximation	% error
100	468	4.00
1000	453.6	0.80

Figure 18

One of the problems with the Monte Carlo method involves the relatively low accuracy attainable. As Fröberg states, " ... with 100 pairs we get a precision of the order of ±5%, and the traditional formulas, for example, Simpson's formula, are much better." [1, p.275] For multiple integrals, however, the smaller error in these other methods becomes substantial as they are repeatedly used for each cross section and then summed. Because of this, Fröberg also states that, " ... it might well be the case that the Monte Carlo method compares favorably, at least if the number of dimensions is ≥ 6." [1, p.275] The best accuracy with the Monte Carlo method can be obtained by using a large number of random points. This is because the error involved decreases with the square root of the number of points, so a decrease in the error to 10% of its present value requires the use of 100 times the present number of random points. [1, p.275]

Another method mentioned previously, the Taylor series approach, is not very useful for multiple integrals. This is due to the complexity of the Taylor series formula for more than a single variable.

Conclusions

Frequently in mathematics, physics, engineering, and other sciences, an integral will arise which can not be readily solved because it either has no antiderivative or has an antiderivative that can be found only after hours or days of effort. In such a situation, an approximation of the integral is useful as long as it is fairly accurate. The preceding sections have presented approximation techniques which can be applied to many integrals. Other methods of approximation exist, but most of these are for special cases and lack the generality to be very applicable over a broad range of functions.

Theoretically, the accuracy for each of the previous techniques is limited only by the amount of time one has to perform the necessary calculations. In the deterministic methods, the accuracy is proportional to the number of calculations, whereas the stochastic methods have an accuracy proportional to the square root of the number of random points generated.

Because of this, the deterministic methods, such as the Newton-Cotes method, should probably be used for single integrals and double integrals. As the number of dimensions of a function increases, however, a stochastic approach should be taken to avoid summing individual errors in repeated calculations. In either case, more calculations will give less error. In order to obtain high accuracy,

these techniques should be performed by computer where less time is required to obtain an answer.

Overall, the preceding sections provide a categorized compilation of general approximation techniques for integrals. This should form a good reference for anyone who encounters a problem integral which can not be solved directly within a reasonable amount of time and effort.

References

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3. Kline, Morris. 1972. Mathematical Thought from Ancient to Modern Times. Oxford University Press. New York.

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5. Sears, Francis W., and Gerhard L. Salinger. 1975. Thermodynamics, Kinetic Theory, and Statistical Thermodynamics. Addison-Wesley Publishing Company, Inc. Reading, Massachusetts.

6. Spiegel, Murray R. 1968. Mathematical Handbook of Formulas and Tables. McGraw-Hill Book Company. New York.

7. Stein, Sherman K. 1982. Calculus and Analytic Geometry. 3rd ed. McGraw-Hill Book Company. New York.

8. Stroud, A. H. 1971. Approximate Calculation of Multiple Integrals. Prentice-Hall, Inc. Englewood Cliffs, New Jersey.

Appendix

The following program estimates the derivatives of a function. Before running this program, the user must modify line 130 which calculates the y value at a specific x value. For the function $y = x^2$, for instance, this line would need to be modified as shown below.

```
130 C(I) = X*X
```

```
  TYPE DERIVATIVE.$1
10 CLS
20 PRINT "THIS PROGRAM ESTIMATES DERIVATIVES."
30 PRINT:PRINT "ENTER LEFT BOUNDARY.":INPUT A
40 PRINT:PRINT "ENTER RIGHT BOUNDARY.":INPUT B
50 PRINT:PRINT "ENTER THE NUMBER OF INTERVALS TO BE USED.":INPUT N
60 H=(B-A)/N
100 DIM C(N+1):DIM D(N)
110 FOR I=1 TO (N+1)
120 X=A + ((I-1)*H)
130 C(I)=
140 NEXT I
150 FOR J=1 TO N
160 D(J)=((C(J+1))-(C(J)))/H
170 PRINT J,D(J)
180 NEXT J
200 PRINT:PRINT "DO YOU NEED THE NEXT HIGHER DERIVATIVE (YES OR NO)?":INPUT A$
210 IF A$="YES" THEN 230
220 END
230 N=N-1
240 FOR I=1 TO (N+1)
250 C(I)=D(I)
260 NEXT I
270 FOR J=1 TO N
280 D(J)=((C(J+1))-(C(J)))/H
290 PRINT J,D(J)
300 NEXT J
310 GOTO 200
```

The following program is an example of the Monte Carlo method for the formula $I \approx \frac{V}{N} \sum_{i=1}^{N} f(\xi_i)$. This program was used to approximate the triple integral on page 30.

```
  TYPE MONTECARLO.$1
10 FOR I=1 TO 100
20 A=RND(100)
30 B=RND(100)
40 C=RND(100)
50 X=(A/100)*3
60 Y=(B/100)*3
70 Z=(C/100)*3
80 D=X*X + Y*Y + Z*Z
90 IF D>9 THEN 110
100 S=S+1
110 NEXT I
120 PRINT "THE APPROXIMATE ANSWER IS",(S/100)*27
130 END
```

The following program estimates an integral of the form $\int_a^b f(x)\,dx$ by using the Newton-Cotes method. Before running the program, the user must modify line 11010 which calculates the y value at a specific x value. Using the function $y = x^2$ as an example, this line would need to be modified as shown below.

```
11010 Y(J) = (X(J))*(X(J))
```

```
  TYPE NCM1.;1
10 CLS
20 PRINT:PRINT "THIS PROGRAM ESTIMATES A SINGLE INTEGRAL BY THE NEWTON-COTES MET
HOD."
30 PRINT:PRINT "ENTER THE FORMULA NUMBER TO BE USED. REMEMBER THAT THE TOTAL NUM
BER OF INTERVALS IN THE X DIRECTION MUST BE DIVISIBLE BY THIS NUMBER."
35 PRINT:PRINT "THE FORMULA NUMBER REPRESENTS THE HIGHEST DEGREE OF THE POLYNOMI
AL TO BE FITTED TO THE CURVE."
40 PRINT:PRINT "WHICH FORMULA NUMBER (1-10)?":INPUT W
50 IF W=0 OR W>10 GOTO 40
60 PRINT:PRINT "ENTER LEFT BOUNDARY.":INPUT A
70 PRINT:PRINT "ENTER RIGHT BOUNDARY.":INPUT B
80 PRINT:PRINT "ENTER THE NUMBER OF INTERVALS TO BE USED IN THE X DIRECTION. THI
S NUMBER MUST BE DIVISIBLE BY THE FORMULA NUMBER CHOSEN.":INPUT N
90 E=N/W
200 DIM X(W+1):DIM Y(W+1)
210 H=(B-A)/N
310 ON W GOSUB 1000,2000,3000,4000,5000,6000,7000,8000,9000,10000
320 Q=L*(B-A)
350 S=Q/E
360 PRINT:PRINT "THE APPROXIMATE ANSWER IS",S
999 END

1000 REM SUBROUTINE ONE
1005 L=0
1010 FOR I=0 TO (E-1)
1020 FOR J=1 TO 2
1030 X(J)=I*W*H +A+ (J-1)*H
1040 GOSUB 11000
1050 NEXT J
1060 K=(Y(1)+Y(2))/2
1070 L=L+K
1080 NEXT I
1999 RETURN
```

```
2000 REM SUBROUTINE TWO
2005 L=0
2010 FOR I=0 TO (E-1)
2020 FOR J=1 TO 3
2030 X(J)=I*W*H +A+ (J-1)*H
2040 GOSUB 11000
2050 NEXT J
2060 K=(Y(1)+4*Y(2)+Y(3))/6
2070 L=L+K
2080 NEXT I
2999 RETURN

3000 REM SUBROUTINE THREE
3005 L=0
3010 FOR I=0 TO (E-1)
3020 FOR J=1 TO 4
3030 X(J)=I*W*H +A+ (J-1)*H
3040 GOSUB 11000
3050 NEXT J
3060 K=(Y(1)+3*Y(2)+3*Y(3)+Y(4))/8
3070 L=L+K
3080 NEXT I
3999 RETURN

4000 REM SUBROUTINE FOUR
4005 L=0
4010 FOR I=0 TO (E-1)
4020 FOR J=1 TO 5
4030 X(J)=I*W*H +A+ (J-1)*H
4040 GOSUB 11000
4050 NEXT J
4060 K=(7*Y(1)+32*Y(2)+12*Y(3)+32*Y(4)+7*Y(5))/90
4070 L=L+K
4080 NEXT I
4999 RETURN
```

```
5000 REM SUBROUTINE FIVE
5005 L=0
5010 FOR I=0 TO (E-1)
5020 FOR J=1 TO 6
5030 X(J)=I*W*H +A+ (J-1)*H
5040 GOSUB 11000
5050 NEXT J
5060 K=(19*Y(1)+75*Y(2)+50*Y(3)+50*Y(4)+75*Y(5)+19*Y(6))/288
5070 L=L+K
5080 NEXT I
5999 RETURN

6000 REM SUBROUTINE SIX
6005 L=0
6010 FOR I=0 TO (E-1)
6020 FOR J=1 TO 7
6030 X(J)=I*W*H +A+ (J-1)*H
6040 GOSUB 11000
6050 NEXT J
6060 K=(41*Y(1)+216*Y(2)+27*Y(3)+272*Y(4)+27*Y(5)+216*Y(6)+41*Y(7))/840
6070 L=L+K
6080 NEXT I
6999 RETURN

7000 REM SUBROUTINE SEVEN
7005 L=0
7010 FOR I=0 TO (E-1)
7020 FOR J=1 TO 8
7030 X(J)=I*W*H +A+ (J-1)*H
7040 GOSUB 11000
7050 NEXT J
7060 K=(751*Y(1)+3577*Y(2)+1323*Y(3)+2989*Y(4)+2989*Y(5)+1323*Y(6)+3577*Y(7)+751
*Y(8))/17280
7070 L=L+K
7080 NEXT I
7999 RETURN
```

```
8000 REM SUBROUTINE EIGHT
8005 L=0
8010 FOR I=0 TO (E-1)
8020 FOR J=1 TO 9
8030 X(J)=I*W*H +A+ (J-1)*H
8040 GOSUB 11000
8050 NEXT J
8060 K=(989*Y(1)+5888*Y(2)-928*Y(3)+10496*Y(4)-4540*Y(5)+10496*Y(6)-928*Y(7)+588
8*Y(8)+989*Y(9))/28350
8070 L=L+K
8080 NEXT I
8999 RETURN

9000 REM SUBROUTINE NINE
9005 L=0
9010 FOR I=0 TO (E-1)
9020 FOR J=1 TO 10
9030 X(J)=I*W*H +A+ (J-1)*H
9040 GOSUB 11000
9050 NEXT J
9060 K=(2857*Y(1)+15741*Y(2)+1080*Y(3)+19344*Y(4)+5778*Y(5)+5778*Y(6)+19344*Y(7)
+1080*Y(8)+15741*Y(9)+2857*Y(10))/89600
9070 L=L+K
9080 NEXT I
9999 RETURN

10000 REM SUBROUTINE TEN
10005 L=0
10010 FOR I=0 TO (E-1)
10020 FOR J=1 TO 11
10030 X(J)=I*W*H +A+ (J-1)*H
10040 GOSUB 11000
10050 NEXT J
10060 K=(16067*Y(1)+106300*Y(2)-48525*Y(3)+272400*Y(4)-260550*Y(5)+427368*Y(6)-2
60550*Y(7)+272400*Y(8)-48525*Y(9)+106300*Y(10)+16067*Y(11))/598752
10070 L=L+K
10080 NEXT I
10999 RETURN

11000 REM FUNCTION SUBROUTINE
11010 Y(J)=
11020 RETURN
```

The following program estimates an integral of the form $\int_a^b \int_c^d f(x,y)\,dy\,dx$ by using the Newton-Cotes method to approximate the area of horizontal cross sections. Before running the program, the user must modify line 11010 which calculates the z values at specific values of x and y. Using the function z = xy as an example, this line would need to be modified as shown below.

```
11010 Z(J) = (X(J))*Y
```

```
  TYPE NCM2.;1
10 CLS
20 PRINT:PRINT "THIS PROGRAM ESTIMATES A DOUBLE INTEGRAL BY REPEATED APPLICATION
 OF THE NEWTON-COTES METHOD OVER A SQUARE OR RECTANGULAR BASE."
30 PRINT:PRINT "ENTER THE FORMULA NUMBER TO BE USED. REMEMBER THAT THE TOTAL NUM
BER OF INTERVALS IN THE X DIRECTION MUST BE DIVISIBLE BY THIS FORMULA NUMBER."
35 PRINT:PRINT "THE FORMULA NUMBER REPRESENTS THE HIGHEST DEGREE OF THE POLYNOMI
AL TO BE FITTED TO THE CURVE."
40 PRINT:PRINT "WHICH FORMULA NUMBER (1-10)?":INPUT W
50 IF W=0 OR W>10 GOTO 40
60 PRINT:PRINT "ENTER LEFT BOUNDARY.":INPUT A
70 PRINT:PRINT "ENTER RIGHT BOUNDARY.":INPUT B
80 PRINT:PRINT "ENTER THE NUMBER OF INTERVALS TO BE USED IN THE X DIRECTION. THI
S NUMBER MUST BE DIVISIBLE BY THE FORMULA NUMBER CHOSEN.":INPUT N
90 E=N/W
100 PRINT:PRINT "ENTER LOWER BOUNDARY.":INPUT C
110 PRINT:PRINT "ENTER UPPER BOUNDARY.":INPUT D
120 PRINT:PRINT "ENTER THE NUMBER OF INTERVALS TO BE USED IN THE Y DIRECTION. TH
ERE ARE NO CONSTRAINTS ON THIS NUMBER.":INPUT M
130 F=(D-C)/M
200 DIM X(W+1):DIM Z(W+1)
210 H=(B-A)/N
300 FOR P=1 TO M
305 Y=C + (P-1)*F
310 ON W GOSUB 1000,2000,3000,4000,5000,6000,7000,8000,9000,10000
320 Q=L*(B-A)
330 R=F*Q
340 S=S+(R/E)
350 NEXT P
360 PRINT:PRINT "THE APPROXIMATE ANSWER IS",S
999 END
```

```
1000 REM SUBROUTINE ONE
1005 L=0
1010 FOR I=0 TO (E-1)
1020 FOR J=1 TO 2
1030 X(J)=I*W*H +A+ (J-1)*H
1040 GOSUB 11000
1050 NEXT J
1060 K=(Z(1)+Z(2))/2
1070 L=L+K
1080 NEXT I
1999 RETURN

2000 REM SUBROUTINE TWO
2005 L=0
2010 FOR I=0 TO (E-1)
2020 FOR J=1 TO 3
2030 X(J)=I*W*H +A+ (J-1)*H
2040 GOSUB 11000
2050 NEXT J
2060 K=(Z(1)+4*Z(2)+Z(3))/6
2070 L=L+K
2080 NEXT I
2999 RETURN

3000 REM SUBROUTINE THREE
3005 L=0
3010 FOR I=0 TO (E-1)
3020 FOR J=1 TO 4
3030 X(J)=I*W*H +A+ (J-1)*H
3040 GOSUB 11000
3050 NEXT J
3060 K=(Z(1)+3*Z(2)+3*Z(3)+Z(4))/8
3070 L=L+K
3080 NEXT I
3999 RETURN
```

```
4000 REM SUBROUTINE FOUR
4005 L=0
4010 FOR I=0 TO (E-1)
4020 FOR J=1 TO 5
4030 X(J)=I*W*H +A+ (J-1)*H
4040 GOSUB 11000
4050 NEXT J
4060 K=(7*Z(1)+32*Z(2)+12*Z(3)+32*Z(4)+7*Z(5))/90
4070 L=L+K
4080 NEXT I
4999 RETURN

5000 REM SUBROUTINE FIVE
5005 L=0
5010 FOR I=0 TO (E-1)
5020 FOR J=1 TO 6
5030 X(J)=I*W*H +A+ (J-1)*H
5040 GOSUB 11000
5050 NEXT J
5060 K=(19*Z(1)+75*Z(2)+50*Z(3)+50*Z(4)+75*Z(5)+19*Z(6))/288
5070 L=L+K
5080 NEXT I
5999 RETURN

6000 REM SUBROUTINE SIX
6005 L=0
6010 FOR I=0 TO (E-1)
6020 FOR J=1 TO 7
6030 X(J)=I*W*H +A+ (J-1)*H
6040 GOSUB 11000
6050 NEXT J
6060 K=(41*Z(1)+216*Z(2)+27*Z(3)+272*Z(4)+27*Z(5)+216*Z(6)+41*Z(7))/840
6070 L=L+K
6080 NEXT I
6999 RETURN
```

```
7000 REM SUBROUTINE SEVEN
7005 L=0
7010 FOR I=0 TO (E-1)
7020 FOR J=1 TO 8
7030 X(J)=I*W*H +A+ (J-1)*H
7040 GOSUB 11000
7050 NEXT J
7060 K=(751*Z(1)+3577*Z(2)+1323*Z(3)+2989*Z(4)+2989*Z(5)+1323*Z(6)+3577*Z(7)+751
*Z(8))/17280
7070 L=L+K
7080 NEXT I
7999 RETURN

8000 REM SUBROUTINE EIGHT
8005 L=0
8010 FOR I=0 TO (E-1)
8020 FOR J=1 TO 9
8030 X(J)=I*W*H +A+ (J-1)*H
8040 GOSUB 11000
8050 NEXT J
8060 K=(989*Z(1)+5888*Z(2)-928*Z(3)+10496*Z(4)-4540*Z(5)+10496*Z(6)-928*Z(7)+588
8*Z(8)+989*Z(9))/28350
8070 L=L+K
8080 NEXT I
8999 RETURN

9000 REM SUBROUTINE NINE
9005 L=0
9010 FOR I=0 TO (E-1)
9020 FOR J=1 TO 10
9030 X(J)=I*W*H +A+ (J-1)*H
9040 GOSUB 11000
9050 NEXT J
9060 K=(2857*Z(1)+15741*Z(2)+1080*Z(3)+19344*Z(4)+5778*Z(5)+5778*Z(6)+19344*Z(7)
+1080*Z(8)+15741*Z(9)+2857*Z(10))/89600
9070 L=L+K
9080 NEXT I
9999 RETURN
```

```
10000 REM SUBROUTINE TEN
10005 L=0
10010 FOR I=0 TO (E-1)
10020 FOR J=1 TO 11
10030 X(J)=I*W*H +A+ (J-1)*H
10040 GOSUB 11000
10050 NEXT J
10060 K=(16067*Z(1)+106300*Z(2)-48525*Z(3)+272400*Z(4)-260550*Z(5)+427368*Z(6)-2
60550*Z(7)+272400*Z(8)-48525*Z(9)+106300*Z(10)+16067*Z(11))/598752
10070 L=L+K
10080 NEXT I
10999 RETURN

11000 REM FUNCTION SUBROUTINE
11010 Z(J)=
11020 RETURN
```

The following program estimates an integral of the form $\int_a^b \int_{y_1=f_1(x)}^{y_2=f_2(x)} f(x,y)\,dy\,dx$ by using the Newton-Cotes method to approximate the area of vertical cross sections. An integral of the form $\int_c^d \int_{x_1=f_1(y)}^{x_2=f_2(y)} f(x,y)\,dx\,dy$ can be estimated with this program by a change of variables which results in an integral of the previous form. Before running the program, lines 306, 307, and 11010 must be modified. Line 306 is the upper boundary in terms of x, line 307 is the lower boundary in terms of x, and line 11010 calculates the z value at specific values of x and y. For the integral $\int_1^4 \int_x^{x^2} xy\,dy\,dx$, these lines would need to be modified as shown below.

```
  306 T = X*X
  307 U = X
11010 Z(J) = X*Y(J)
```

```
  TYPE NCM3.;1
10 CLS
20 PRINT:PRINT "THIS PROGRAM ESTIMATES A DOUBLE INTEGRAL BY REPEATED APPLICATION
 OF THE NEWTON-COTES METHOD OVER AN IRREGULAR BASE."
30 PRINT:PRINT "ENTER THE FORMULA NUMBER TO BE USED. REMEMBER THAT THE TOTAL NUM
BER OF INTERVALS IN THE Y DIRECTION MUST BE DIVISIBLE BY THIS NUMBER."
35 PRINT:PRINT "THE FORMULA NUMBER REPRESENTS THE HIGHEST DEGREE OF THE POLYNOMI
AL TO BE FITTED TO THE CURVE."
40 PRINT:PRINT "WHICH FORMULA NUMBER (1-10)?":INPUT W
50 IF W=0 OR W>10 GOTO 40
60 PRINT:PRINT "ENTER LEFT BOUNDARY.":INPUT A
70 PRINT:PRINT "ENTER RIGHT BOUNDARY.":INPUT B
80 PRINT:PRINT "ENTER THE NUMBER OF INTERVALS TO BE USED IN THE X DIRECTION. THE
RE ARE NO CONSTRAINTS ON THIS NUMBER.":INPUT N
90 E=(B-A)/N
120 PRINT:PRINT "ENTER THE NUMBER OF INTERVALS TO BE USED IN THE Y DIRECTION. TH
IS NUMBER MUST BE DIVISIBLE BY THE FORMULA NUMBER CHOSEN.":INPUT M
130 F=M/W
200 DIM Y(W+1):DIM Z(W+1)
300 FOR P=1 TO N
305 X=A + (P-1)*E
306 T=
307 U=
308 H=(T-U)/M
310 ON W GOSUB 1000,2000,3000,4000,5000,6000,7000,8000,9000,10000
320 Q=L*(T-U)
330 R=E*Q
340 S=S+(R/F)
350 NEXT P
360 PRINT:PRINT "THE APPROXIMATE ANSWER IS",S
999 END
```

```
1000 REM SUBROUTINE ONE
1005 L=0
1010 FOR I=0 TO (F-1)
1020 FOR J=1 TO 2
1030 Y(J)=I*W*H +U+ (J-1)*H
1040 GOSUB 11000
1050 NEXT J
1060 K=(Z(1)+Z(2))/2
1070 L=L+K
1080 NEXT I
1999 RETURN

2000 REM SUBROUTINE TWO
2005 L=0
2010 FOR I=0 TO (F-1)
2020 FOR J=1 TO 3
2030 Y(J)=I*W*H +U+ (J-1)*H
2040 GOSUB 11000
2050 NEXT J
2060 K=(Z(1)+4*Z(2)+Z(3))/6
2070 L=L+K
2080 NEXT I
2999 RETURN

3000 REM SUBROUTINE THREE
3005 L=0
3010 FOR I=0 TO (F-1)
3020 FOR J=1 TO 4
3030 Y(J)=I*W*H +U+ (J-1)*H
3040 GOSUB 11000
3050 NEXT J
3060 K=(Z(1)+3*Z(2)+3*Z(3)+Z(4))/8
3070 L=L+K
3080 NEXT I
3999 RETURN
```

```
4000 REM SUBROUTINE FOUR
4005 L=0
4010 FOR I=0 TO (F-1)
4020 FOR J=1 TO 5
4030 Y(J)=I*W*H +U+ (J-1)*H
4040 GOSUB 11000
4050 NEXT J
4060 K=(7*Z(1)+32*Z(2)+12*Z(3)+32*Z(4)+7*Z(5))/90
4070 L=L+K
4080 NEXT I
4999 RETURN

5000 REM SUBROUTINE FIVE
5005 L=0
5010 FOR I=0 TO (F-1)
5020 FOR J=1 TO 6
5030 Y(J)=I*W*H +U+ (J-1)*H
5040 GOSUB 11000
5050 NEXT J
5060 K=(19*Z(1)+75*Z(2)+50*Z(3)+50*Z(4)+75*Z(5)+19*Z(6))/288
5070 L=L+K
5080 NEXT I
5999 RETURN

6000 REM SUBROUTINE SIX
6005 L=0
6010 FOR I=0 TO (F-1)
6020 FOR J=1 TO 7
6030 Y(J)=I*W*H +U+ (J-1)*H
6040 GOSUB 11000
6050 NEXT J
6060 K=(41*Z(1)+216*Z(2)+27*Z(3)+272*Z(4)+27*Z(5)+216*Z(6)+41*Z(7))/840
6070 L=L+K
6080 NEXT I
6999 RETURN
```

```
7000 REM SUBROUTINE SEVEN
7005 L=0
7010 FOR I=0 TO (F-1)
7020 FOR J=1 TO 8
7030 Y(J)=I*W*H +U+ (J-1)*H
7040 GOSUB 11000
7050 NEXT J
7060 K=(751*Z(1)+3577*Z(2)+1323*Z(3)+2989*Z(4)+2989*Z(5)+1323*Z(6)+3577*Z(7)+751
*Z(8))/17280
7070 L=L+K
7080 NEXT I
7999 RETURN

8000 REM SUBROUTINE EIGHT
8005 L=0
8010 FOR I=0 TO (F-1)
8020 FOR J=1 TO 9
8030 Y(J)=I*W*H +U+ (J-1)*H
8040 GOSUB 11000
8050 NEXT J
8060 K=(989*Z(1)+5888*Z(2)-928*Z(3)+10496*Z(4)-4540*Z(5)+10496*Z(6)-928*Z(7)+588
8*Z(8)+989*Z(9))/28350
8070 L=L+K
8080 NEXT I
8999 RETURN

9000 REM SUBROUTINE NINE
9005 L=0
9010 FOR I=0 TO (F-1)
9020 FOR J=1 TO 10
9030 Y(J)=I*W*H +U+ (J-1)*H
9040 GOSUB 11000
9050 NEXT J
9060 K=(2857*Z(1)+15741*Z(2)+1080*Z(3)+19344*Z(4)+5778*Z(5)+5778*Z(6)+19344*Z(7)
+1080*Z(8)+15741*Z(9)+2857*Z(10))/89600
9070 L=L+K
9080 NEXT I
9999 RETURN
```

```
10000 REM SUBROUTINE TEN
10005 L=0
10010 FOR I=0 TO (F-1)
10020 FOR J=1 TO 11
10030 Y(J)=I*W*H +U+ (J-1)*H
10040 GOSUB 11000
10050 NEXT J
10060 K=(16067*Z(1)+106300*Z(2)-48525*Z(3)+272400*Z(4)-260550*Z(5)+427368*Z(6)-2
60550*Z(7)+272400*Z(8)-48525*Z(9)+106300*Z(10)+16067*Z(11))/598752
10070 L=L+K
10080 NEXT I
10999 RETURN

11000 REM FUNCTION SUBROUTINE
11010 Z(J)=
11020 RETURN
```